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Classification of Voronoi and Delone tiles of quasicrystals: III. Decagonal acceptance window of any size

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Abstract

This paper is the last of a series of three articles presenting a classification of Vornoi and Delone tilings determined by point sets $\Sigma(\Omega)$ ('quasicrystals'), built by the standard projection of the root lattice of type A_4 to a two-dimensional plane spanned by the roots of the Coxeter group H_2 (dihedral group of order 10). The acceptance window Ω for $\Sigma(\Omega)$ in the present paper is a regular decagon of any radius $0 < r < \infty$. There are 14 distinct VT sets of Voronoi tiles and 6 sets DT of Delone tiles, up to a uniform scaling by the factor τ^k, $\tau = \frac{1}{2}(1+\sqrt{5})$ and $k \in \mathbb{Z}$. The number of Voronoi tiles in different quasicrystal tilings varies between 3 and 12. Similarly, the number of Delone tiles is varying between 4 and 6. There are 7 VT sets of the 'generic' type and 7 of the 'singular' type. The latter occur for seven precise values of the radius of the acceptance window. Quasicrystals with acceptance windows with radii in between these values have constant VT sets, only the relative densities and arrangement of the tiles in the tilings change. Similarly, we distinguish singular and generic sets DT of Delone tiles.

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1. Introduction

The present paper is the last one in a series [5, 6] where we describe sets VT and DT of Voronoi and Delone tiles which appear in tilings of certain standard quasicrystal-like point sets. The three cases differ by the shape of their acceptance windows: rhombus in [5], disc in [6] and regular decagon here.

The motivation for studying such problems and the general method for acceptance windows of any shape, together with some preliminaries indispensable for understanding and using our method, are presented in [5]. We introduce a method which reduces the search for distinct Voronoi and Delone tiles in a quasicrystal to a finite problem. In addition, one also finds there the solution for equilateral rhombic acceptance window oriented along the roots of the Coxeter group H_2 (the dihedral group of order 10).

In [6] and this paper, the method of [5] is used to determine the tiling sets VT and DT for the circular and decagonal acceptance windows, respectively. The two papers are self-contained as far as the understanding of the results is concerned. In order to grasp the comments of the procedure to achieve the results, [5] needs to be consulted.

The quasicrystals of the three papers are infinite two-dimensional point sets constructed by the cut-and-project method. More precisely, points of the root lattice of the simple Lie algebra A_4 are projected onto two complementary two-dimensional subspaces invariant under the action of the Coxeter group H_2 which is a subgroup of the Weyl group of A_4. Under such projections, the integer coordinates of the A_4-lattice points acquire the irrationality given by the solutions of $x^2 = x + 1$, i.e. the golden ratio $\tau = \frac{1}{2}(1 + \sqrt{5})$. For more details see for example [5]. Members of such a family are called H_2-quasicrystals. They differ by the shape and size of their acceptance window.

A motivation for studying the present case, i.e. quasicrystals with decagonal acceptance window, can be brought up by asking deceivingly simple question: which of the uncountably many non-isomorphic tilings of the family of H_2-quasicrystals are the simplest? An answer to the question can be given only after one specifies what the superlative 'simplest' actually means. Let us point out some obvious possibilities first.

The 'simplest' can be taken to mean the smallest number of distinct shapes of tiles. In that case the clear winners are the well-known Penrose tilings with two types of tiles only. However, the Penrose tilings can be taken as a member of our H_2-family of quasicrystals only if we introduce the composition of five different pentagonal acceptance windows, see [4].

Secondly, the 'simplest' can be understood as the one with the simplest acceptance window. In that case, the disc is certainly much simpler than the composition of five windows required for the Penrose tiling. Disc window is also simple for any actual computing of quasicrystal points.

However, if the simplicity should mean a conceptual simplicity of the definition of the quasicrystal, acceptance window as a regular decagon inscribed in a unit circle has a fair claim to be the simplest. Indeed, such decagon is a convex hull of the roots of the Coxeter group H_2, therefore the corresponding quasicrystal is completely defined by the properties of the group. Precisely for that reason, quasicrystals of this type for the groups H_2, H_3 and H_4 are called canonical. A fraction of a canonical quasicrystal is shown in figure 1 of [3].

The problem solved in this paper is somewhat more general than the canonical quasicrystal case. We determine all the sets VT and DT of Voronoi and Delone tiles corresponding to tilings of H_2-quasicrystals with the acceptance window being a regular decagon, oriented along the H_2-roots of any radius $\tau^{-1} < r \leqslant 1$. For decagons of radii outside of these bounds, the same sets VT and DT occur, only scaled by a suitable power of τ.

Specifically, we determine that there are 14 distinct VT sets of Voronoi tiles, seven of them being singular, each appearing for a single value of r; and there are six sets DT of Delone tiles with three of them being singular. The Voronoi tiling of the canonical H_2-quasicrystal is the most simple in that its VT and DT set contain three tiles only. Leaving aside the specific case of the Penrose tilings, it appears that the canonical quasicrystals are the most simple with respect to their tiling sets.

2. Preliminaries

Let us now introduce the definition of cut-and-project quasicrystals studied in this paper. Since our aim is to apply the method of [5] to a specific type of quasicrystal models, we recall only the necessary facts. Cut-and-project quasicrystals may be defined in a much more general setting; this is however out of the scope of this work. For general presentation we refer to [5], where one also finds references to related articles.

The ring of integers in the quadratic field $\mathbb{Q}(\sqrt{5})$ is the set $\mathbb{Z}[\tau] = \mathbb{Z} + \mathbb{Z}\tau$, where $\tau = \frac{1}{2}(1+\sqrt{5})$ is the golden ratio. Its algebraic conjugate is denoted by $\tau' = \frac{1}{2}(1-\sqrt{5})$. We denote $\Delta = \sqrt{\tau+2}$. We make use of the Galois automorphism of $\mathbb{Q}(\sqrt{5})$:

$$x = a + b\tau \quad \mapsto \quad x' = a + b\tau'.$$

The root system H_2 can be represented in the complex plane by the tenth roots of unity, ξ^j, $j = 0, \dots, 9$. The simple roots of this system are $\alpha_1 = \xi^0 = 1$ and $\alpha_2 = \xi^4$. All other roots can be expressed in the basis α_1, α_2 with only non-negative or only non-positive coefficients,

$$\xi^0 = \alpha_1, \qquad \xi^1 = \tau\alpha_1 + \alpha_2, \qquad \xi^2 = \tau\alpha_1 + \tau\alpha_2,$$
$$\xi^3 = \alpha_1 + \tau\alpha_2, \qquad \xi^4 = \alpha_2, \qquad \xi^{5+i} = -\xi^i \quad \text{for} \quad i = 0, \dots, 4.$$

Using the above relations, it is obvious that the $\mathbb{Z}$-span of the root system H_2 is in fact a $\mathbb{Z}[\tau]$-module

$$M = \mathbb{Z}[\tau]\alpha_1 + \mathbb{Z}[\tau]\alpha_2.$$

One defines a 'star-map' on M by

$$M \ni x = x_1\alpha_1 + x_2\alpha_2, \qquad x_1, x_2 \in \mathbb{Z}[\tau] \quad \mapsto \quad x^* = x_1'\alpha_1^* + x_2'\alpha_2^*,$$

where $\alpha_1^* = \alpha_1 = 1$ and $\alpha_2^* = \alpha_2^2 = \xi^8$. It is easy to show that the star-map preserves the root system H_2 and is semi-linear with respect to the Galois conjugation ', i.e. $(ux+y)^* = u'x^* + y^*$ for $u \in \mathbb{Z}[\tau]$, $x, y \in M$.

With the above notations, one defines a cut-and-project quasicrystal as the set

$$\Sigma(\Omega) = \{x \in M \mid x^* \in \Omega\},$$

where Ω is a bounded set satisfying $\overline{\Omega^\circ} = \overline{\Omega}$ and is called the acceptance window. The semi-linearity of the star-map implies an important scaling property of the quasicrystal, namely

$$\tau\Sigma(\Omega) = \Sigma(\tau'\Omega). \tag{1}$$

This property allows us to restrict the considerations only to certain sizes of the acceptance window.

Cut-and-project quasicrystals have many interesting properties. First, they are uniformly discrete, relatively dense sets (Delone property), they are almost lattices, etc. The properties of the cut-and-project quasicrystal depend on the choice of the acceptance window Ω. It is usual to require Ω closed or open. The singularities in the cut-and-project quasicrystal caused by the boundary of Ω can be avoided if we impose $\partial\Omega \cap M = \emptyset$. In such a case, the quasicrystal is repetitive, i.e. every finite configuration appears in $\Sigma(\Omega)$ with non-zero density, and thus also every tile in the corresponding Voronoi, resp. Delone tiling has non-zero density.

The specific cut-and-project scheme considered here is chosen for it produces quasicrystal models with 10-fold symmetry (if the acceptance window has it). If moreover Ω is chosen convex, the quasicrystal has abundance of scaling symmetries [2]. The symmetries of the cut-and-project quasicrystals are consequence of the symmetries of the $\mathbb{Z}[\tau]$-module M, which are described for example in [1].

In this paper, we consider the acceptance window Ω to be a regular decagon of any size. As a consequence of (1), we can assume that the decagon is inscribed in a circle of radius within the range $(\tau^{-1}, 1]$.

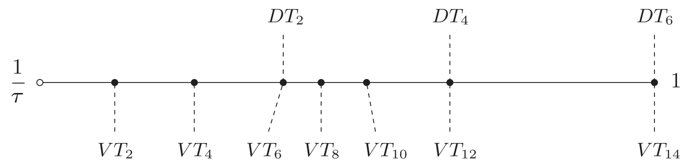


Figure 1. The figure shows the range $(1/\tau, 1\rangle$ of radius of the decagonal acceptance window r divided by singular cases. The figure is drawn in scale. Between two Voronoi/Delone singular cases the set of Voronoi/Delone tiles in tiling does not change.

Table 1. The table shows cases of quasicrystals with decagonal acceptance windows according to sets of Voronoi and Delone tiles. There are 14 classes of quasicrystals VT_m, $m = 1, \ldots, 14$, which have different Voronoi tiles. In the second column there are a number of Voronoi shapes in Voronoi tiling. Even cases are singular and they are represented by quasicrystals with specific size of window, which is denoted in the middle column. On the other hand, there are six classes of quasicrystals DT_m, $m = 1, \ldots, 6$ with decagonal acceptance window, which have different Delone tiles. In the fourth column there are a number of Delone shapes.

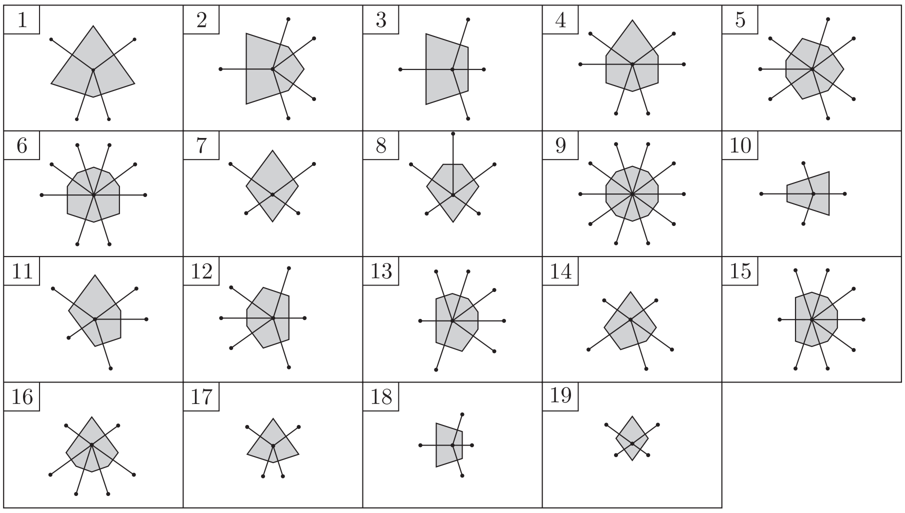
VT_1	10			
VT_2	9	$(\tau + 2)/5 \doteq 0.723\,606\,80$		
VT_3	10		5	DT_1
VT_4	9	$4 - 2\tau \doteq 0.763\,932\,02$		
VT_5	10			
VT_6	6	$\tau/2 \doteq 0.809\,016\,99$	5	DT_2
VT_7	12			
VT_8	10	$(9 - 3\tau)/5 \doteq 0.829\,179\,61$		
VT_9	10		6	DT_3
VT_{10}	9	$3\tau - 4 \doteq 0.854\,101\,97$		
VT_{11}	9			
VT_{12}	5	$(4\tau - 2)/5 \doteq 0.894\,427\,19$	4	DT_4
VT_{13}	6		6	DT_5
VT_{14}	3	1	4	DT_6

3. Results

Let Ω be a decagon given as the convex hull $\Omega = \langle\{r\xi^j + s \mid j = 0, \ldots, 9\}\rangle$, where $r \in (\tau^{-1}, 1]$ and $s \in \mathbb{C}$. We focus on the case $\partial(\Omega) \cap M = \emptyset$. Some remarks about the special situation when $\partial\Omega \cap M \neq \emptyset$, i.e. some tiles appear with zero density, are given in section 4.

Using our method it turns out that for the description of the set VT of Voronoi tiles and DT of Delone tiles in the tiling of $\Sigma(\Omega)$, it is necessary to divide the range $(\tau^{-1}, 1]$ of sizes r of Ω into several subintervals, as shown in figure 1. There are seven subintervals for the Voronoi tiling and three subintervals for the Delone tiling. This corresponds to the fact that there are 14 different sets $\mathrm{VT}_1, \ldots \mathrm{VT}_{14}$, where the even indices give the sets of Voronoi tiles for a singular value of r. Similarly, there are six different sets $\mathrm{DT}_1, \ldots \mathrm{DT}_6$, where the even indices give singular Delone tilings. The singular values of r, together with the number of tiles in the corresponding VT or DT set are presented in table 1.

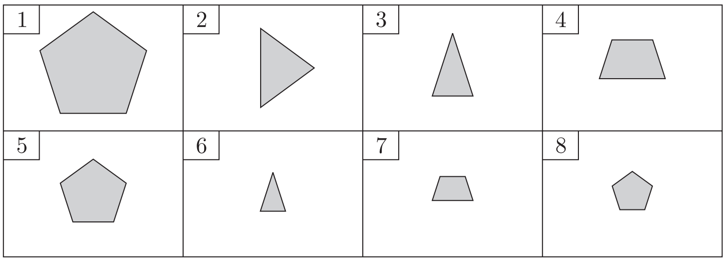
Table 2. The 19 tiles shown in the top table comprise the complete set of Voronoi tiles encountered in all quasicrystals with decagonal acceptance window. Shapes and relative sizes of the tiles are maintained. Also shown are the points of quasicrystal which define the tile. For a fixed radius r of the decagon only a subset VT_m of tiles is present in the Voronoi tiling. The entries at the intersection of a column k and a row VT_m indicate the presence of the tile number k in the set VT_m. With each tile there are at most 20 differently oriented copies in the tiling according to the dihedral group H_{20}. Tiles which are itself symmetric under some subgroup of H_{20} appear in smaller number. For more details see the text.



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
VT_1	•	•	•	•	•	•	•	•	•	•									
VT_2		•	•	•	•	•	•	•	•	•									
VT_3		•	•	•	•	•	•	•	•	•	•								
VT_4			•	•	•	•	•	•	•	•	•								
VT_5			•	•	•	•	•	•	•	•	•	•							
VT_6						•	•	•	•	•		•							
VT_7						•	•	•	•	•		•	•	•	•	•	•	•	
VT_8							•	•	•	•		•	•	•	•	•	•	•	
VT_9							•	•	•	•		•	•	•	•	•	•	•	
VT_{10}							•	•	•	•				•	•	•	•	•	
VT_{11}							•	•	•	•				•	•	•	•	•	
VT_{12}									•	•					•	•	•	•	
VT_{13}									•	•					•	•	•	•	•
VT_{14}																	•	•	•

The sets of Voronoi and Delone tiles are described in tables 2 and 3, respectively. In the upper parts of the tables the tiles are drawn. The bottom table shows which tiles belong to a given set VT_i or DT_i. Note that some shapes of tiles appear in several different sizes.

Table 3. The eight tiles shown in the top table comprise the complete set of Delone tiles encountered in all quasicrystals with decagonal acceptance window. Shapes and relative sizes of the tiles are maintained. For a radius r of the decagon only a subset DT_m of tiles is present in the Delone tiling. The entries at the intersection of a column k and a row DT_m indicate the presence of the tile number k in the set DT_m. With each tile there are at most 20 differently oriented copies in the tiling according to the dihedral group H_{20}. Tiles which are itself symmetric under some subgroup of H_{20} appear in smaller number. For more details see the text.



	1	2	3	4	5	6	7	8
DT_1	•	•	•	•	•			
DT_2	•	•	•	•	•			
DT_3	•	•	•	•	•	•		
DT_4			•	•	•	•		
DT_5			•	•	•	•	•	•
DT_6					•	•	•	•

Just like the circular acceptance window discussed in [6], the decagonal acceptance window is symmetric with respect to the transformations of the dihedral group of order 20, i.e. H_{20}. Since the $\mathbb{Z}[\tau]$-module M has the same symmetry, each Voronoi and also Delone tile may appear in 20 orientations in the quasicrystal. However, certain tiles among those given in tables 2 and 3 are invariant under some of the transformations of H_{20}.

As it was explained in [5], we can determine the type of the Voronoi tile of a chosen point $x \in \Sigma(\Omega)$ according to the position of its star-map image x^* in the acceptance window. In fact, the decagonal acceptance window splits into a finite number of regions. Points having their star-map image in one region have the same type of the Voronoi tile. The volume of the region is proportional to the density of the corresponding tile in the Voronoi tiling. The division of the decagonal acceptance window is shown in figures 2–4. Due to the mentioned symmetries, we show only a section of angle $2\pi/10$, which characterizes the division of the entire decagon. The regions on figures are marked with numbers. The numbering of regions corresponds to the numbering of types of tiles in table 2.

The division of the acceptance window changes with changing size r. Within a subinterval of $r \in (\tau^{-1}, 1]$ which corresponds to one non-singular case, the arrangement of the regions is the same, but the relative sizes (densities of tiles) change.

With decreasing r, we arrive at a singular value of r. This corresponds to a size of the acceptance decagon, where some of the regions in the division disappear, as is illustrated in figure 5. The region reduces to a point or a line segment, which is marked bold in figures 2–4.

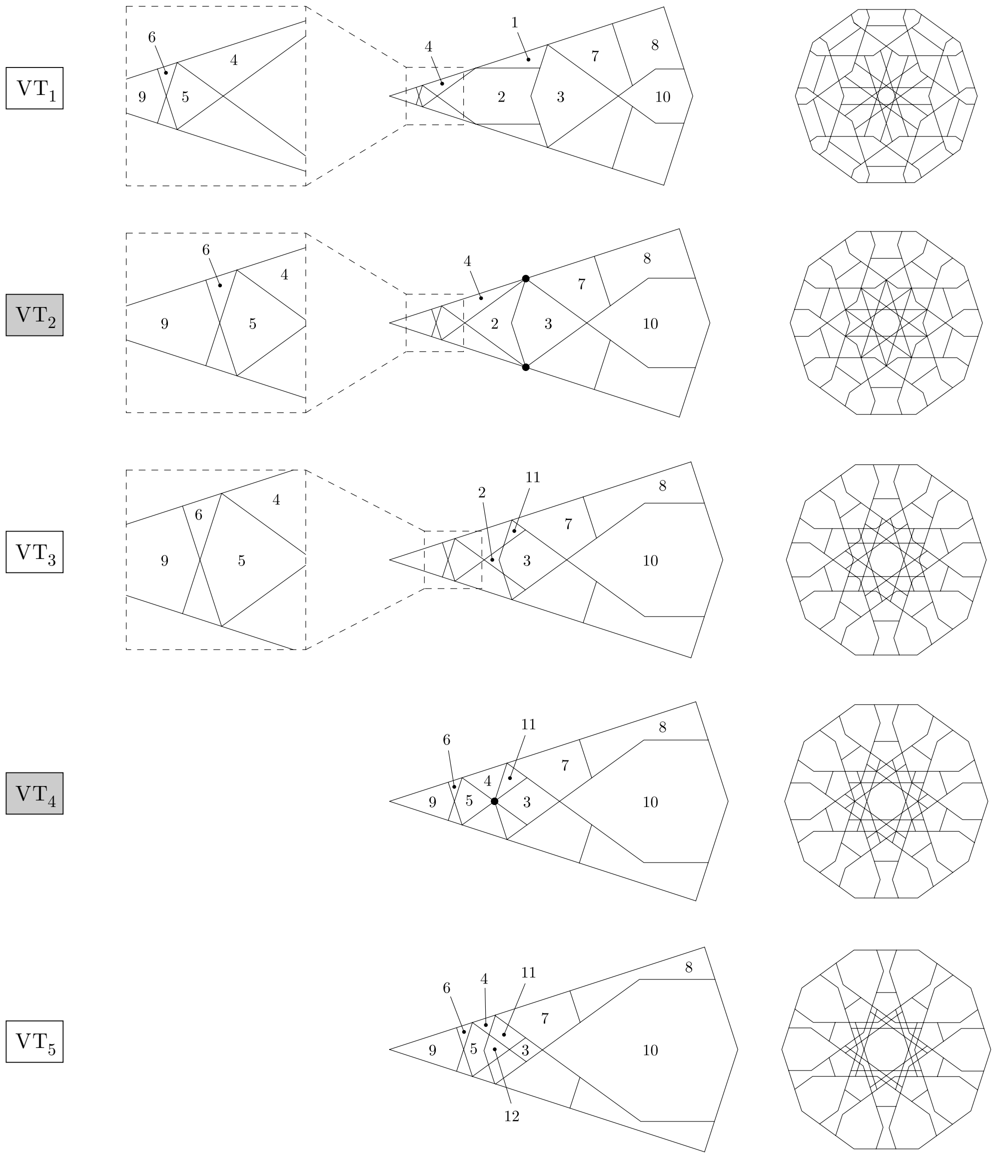


Figure 2. Division of decagonal acceptance window (part 1). Each region in the acceptance window corresponds to a different Voronoi tile. The numbers which denote these regions are the numbers of the corresponding Voronoi tiles from table 2. The parts of the window that correspond to tiles with zero density are marked in bold.

Since we have assumed that the boundary $\partial\Omega$ has an empty intersection with M, the point, resp. line segment also does not contain any element of M, and thus it does not generate a Voronoi tile with zero density.

An example of the most simple Voronoi and Delone tilings is shown in figure 6. It corresponds to the case of decagonal acceptance window with $r = 1$, i.e. to the singular cases VT_{14} and DT_6. Thus according to tables 2 and 3, there are only three types of Voronoi tiles and four types of Delone tiles. Each of the tiles has a non-zero density. For that

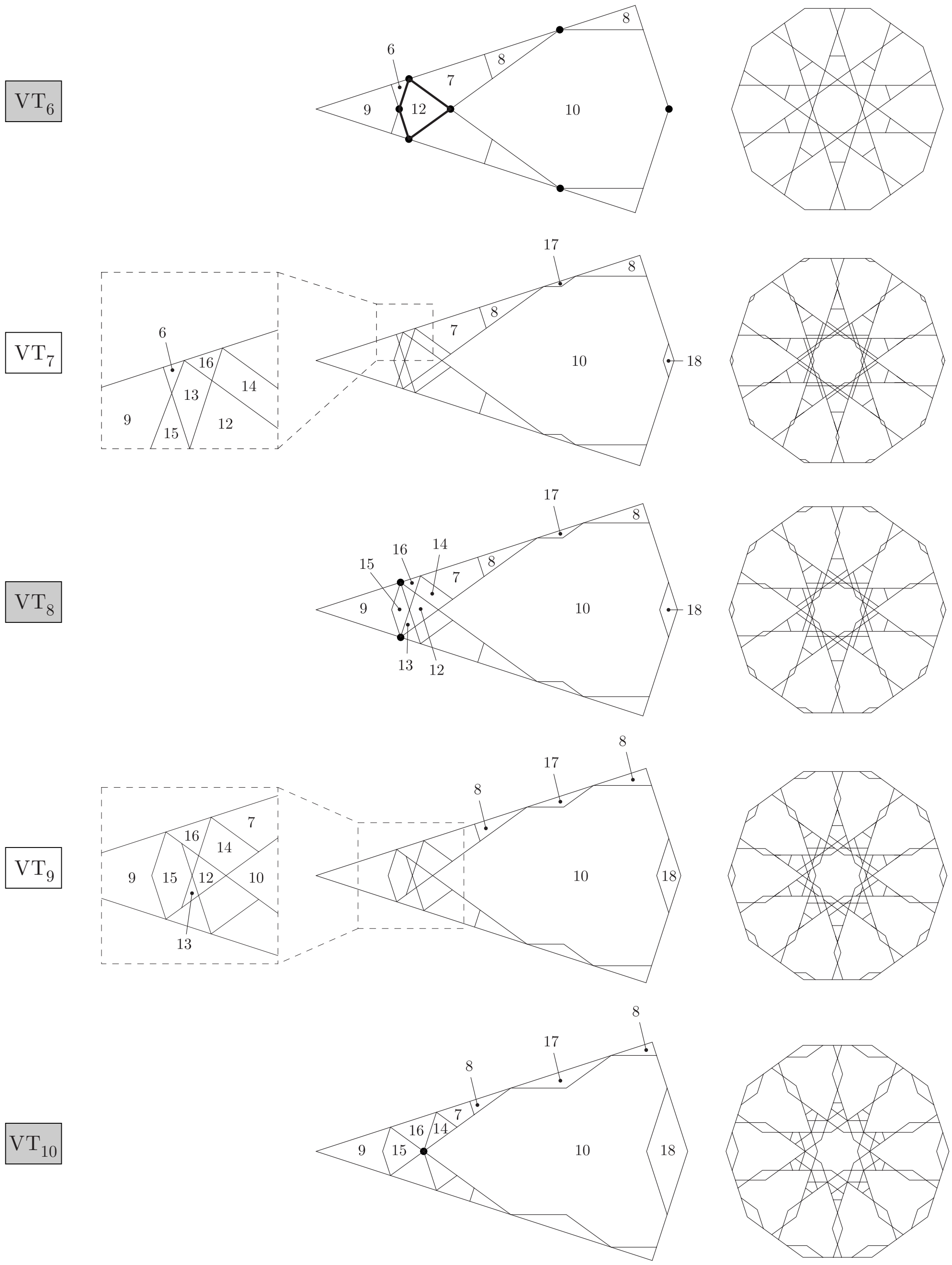


Figure 3. Division of decagonal acceptance window (part 2). Each region in the acceptance window corresponds to a different Voronoi tile. The numbers which denote these regions are the numbers of the corresponding Voronoi tiles from table 2. The parts of the window that correspond to tiles with zero density are marked in bold.

we needed that the intersection of the boundary of Ω with the $\mathbb{Z}[\tau]$-module M is empty. For the size $r = 1$ this can be reached only if the acceptance window is centred at a

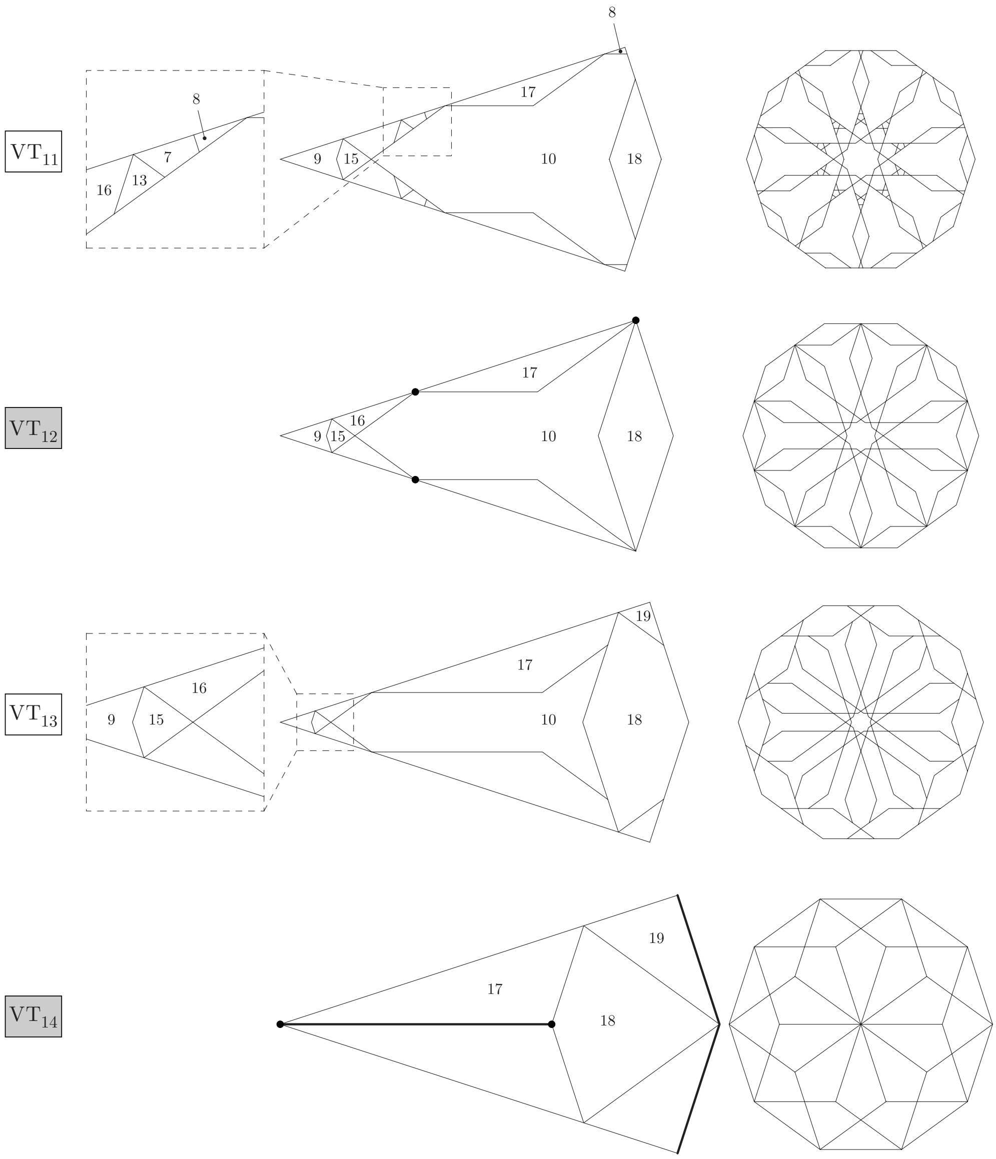


Figure 4. Division of decagonal acceptance window (part 3). Each region in the acceptance window corresponds to a different Voronoi tile. The numbers which denote these regions are the numbers of the corresponding Voronoi tiles from table 2. The parts of the window that correspond to tiles with zero density are marked in bold. Note that the case VT_{14} is the canonical quasicrystal with $r = 1$.

point not belonging to M. Thus, the corresponding tiling does not reveal a global 10-fold symmetry.

Another example of a tiling in which every tile has a non-zero density is found in figure 7. Here we consider as the acceptance window a decagon of radius $r = \tau/\Delta + 1/\tau^6$, which corresponds to the non-singular cases VT_{13} and DT_5. Here we have $\partial\Omega \cap M = \emptyset$ even if the decagon is centred at the origin. Therefore, the tiling has a global 10-fold symmetry.

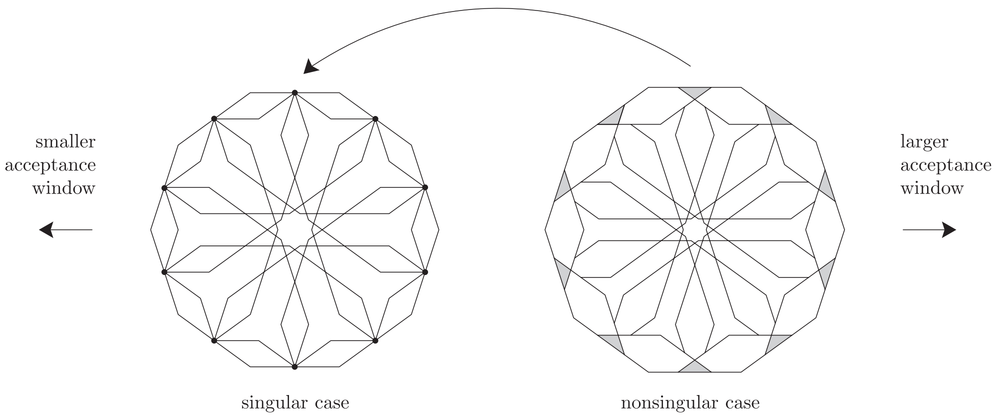


Figure 5. Acceptance window of a singular and non-singular case.

4. Tiles with zero density

So far we have studied the case that the boundary $\partial\Omega$ of acceptance window Ω has an empty intersection with the $\mathbb{Z}[\tau]$-module M. Thus, every tile appears with a non-zero density. Relaxing the condition $\partial\Omega \cap M = \emptyset$ may not be interesting for physics, because the resulting tilings are no longer repetitive and do not belong to the same local isomorphism class as the non-singular ones, although the acceptance window has the same shape and size. However, for completeness, we explain the peculiarities which may occur in this case. Either the Voronoi or Delone tiling of the set $\Sigma(\Omega)$ contains a tile which appears only finitely many times, or even there are some tiles which appear infinitely many times, but still with vanishing density. This phenomena depends on the cardinality of the intersection of the boundary $\partial\Omega$ with the $\mathbb{Z}[\tau]$-module M. For the case of the circular acceptance window treated in [6] the intersection was always finite, cf [7]. In contrast, the boundary of the decagonal acceptance window may contain infinitely many points of M.

If the intersection $\partial\Omega \cap M$ is non-empty, the boundaries of the regions in the division of Ω contain points of M. Thus for certain values of r, there occur some Voronoi tiles which have density 0 in the tiling. The star map images of points with such exceptional tiles lie on the bold marked points/line segments in figures 2–4.

As an example of what may happen, let us consider the decagonal acceptance window inscribed in a unit circle ($r = 1$), now centred at the origin, i.e. again the singular cases VT_{14} and DT_6. The corresponding Voronoi and Delone tilings are shown in figure 8. Clearly, the line segments marked bold in figure 4 contain infinitely many points of M, thus the Voronoi and Delone tilings contain tiles which appear infinitely many times, but with zero density. These tiles are arranged along the axes of 10-fold rotation symmetry.

5. Concluding remarks

There are a number of observations one can make concerning VT and DT tile sets for the three shapes of quasicrystal acceptance windows we have solved in [5, 6] and in this paper.

- Delone tilings are considerably simpler than Voronoi ones, although their determination required a classification of fans of Voronoi tiles. Our method can be directly used, without serious modification, for studying other clusters of tiles.

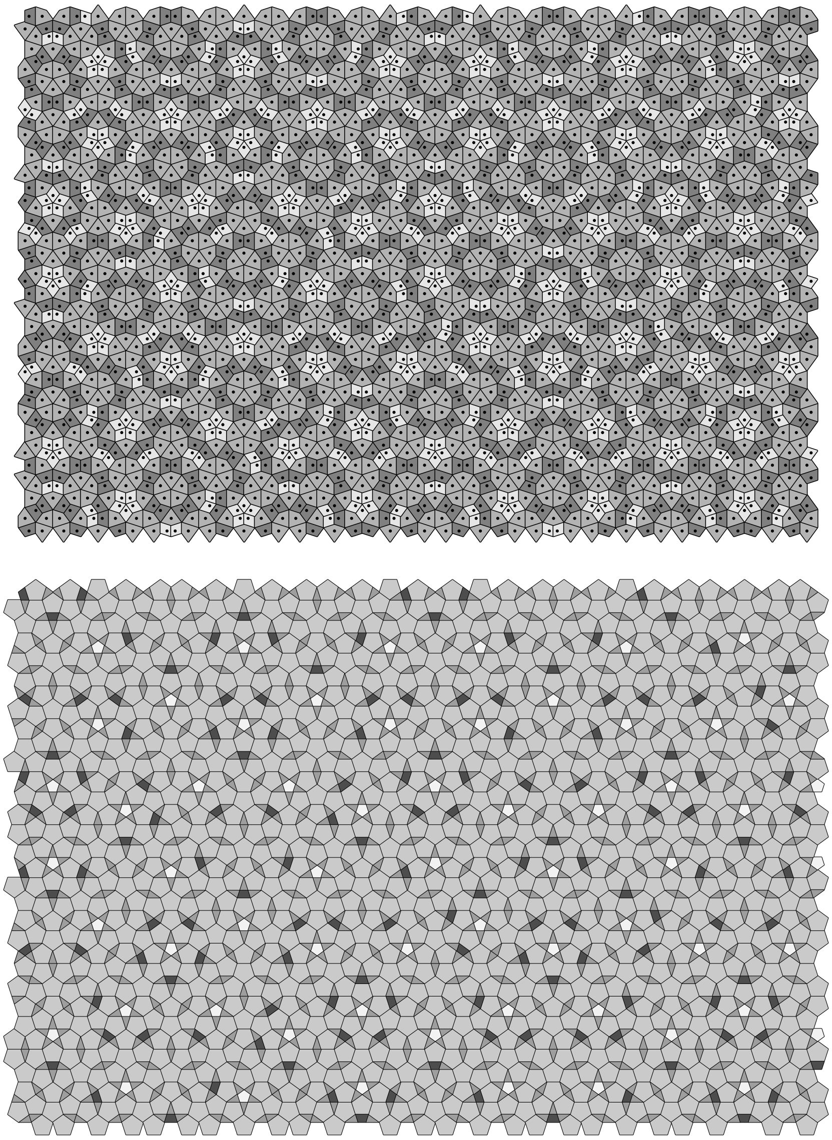


Figure 6. Voronoi and Delone tilings of a quasicrystal with decagonal acceptance window for the most simple cases VT_{14} and DT_6, with $\partial\Omega \cap M = \emptyset$. The tiling has only three types of Voronoi tiles and four types of Delone tiles.

- In all cases we have studied there are only four different shapes of Delone tiles; a DT set may contain several scaled copies of one shape.
- From our study one can possibly learn how the VT and DT sets are modified during continuous changes of the acceptance window. How much the acceptance window can be deformed while preserving the shapes of tiles? In particular, what class of quasicrystals preserves the four shapes of Delone tiles?
- There are other shapes of acceptance windows which would be interesting and not any more difficult to study. Most notably regular pentagons and decagons that are not regular,

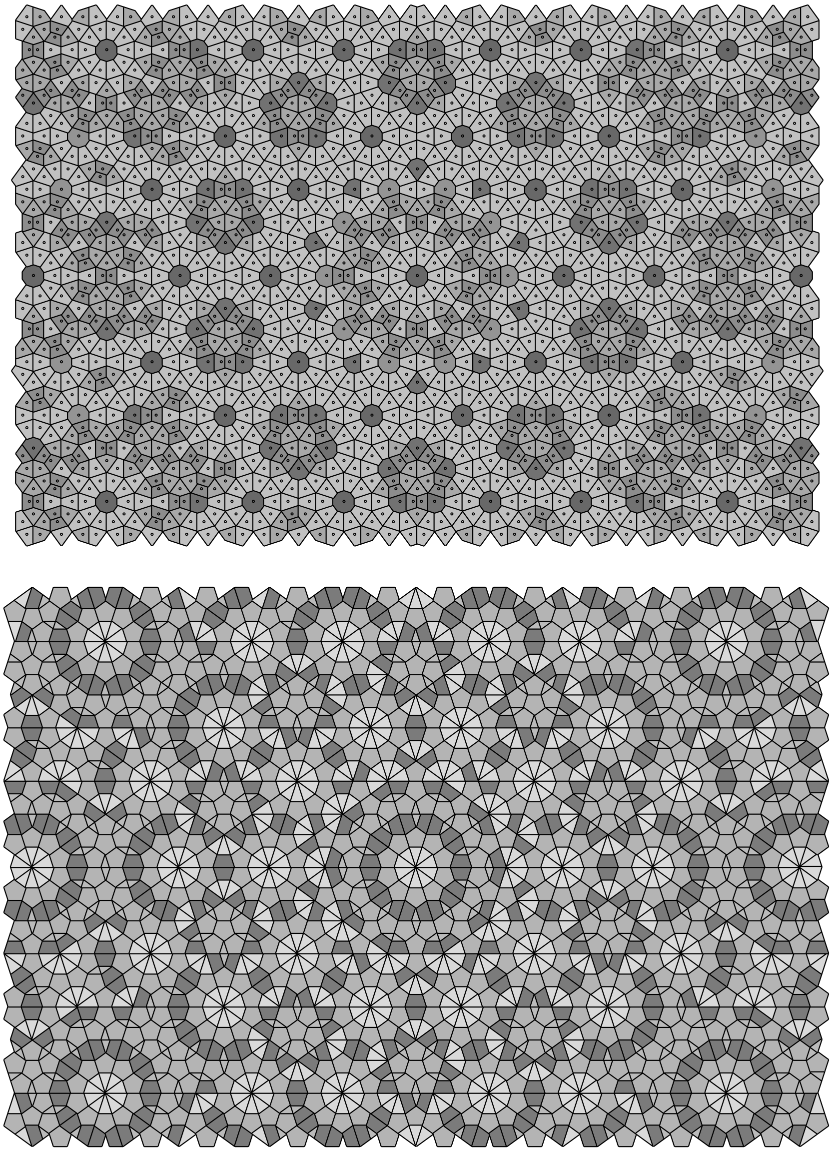


Figure 7. Voronoi and Delone tilings of a quasicrystal with decagonal acceptance window centred at the origin for cases VT_{13} and DT_5. The radius of the acceptance window is $r = \tau/\Delta + 1/\tau^6$. This is a non-singular case.

but still generated by H_2-reflections of a single point, or pentagons and decagons not oriented as the roots of H_2.

- We do not expect additional difficulties if the method is extended to tilings of two-dimensional quasicrystals with local symmetries of dihedral groups of higher orders.
- In three dimensions, a similar study appears prohibitively laborious, the large number of tiles in VT and DT sets being the least of the obstacles. Determination of some singular radii of acceptance windows certainly would be possible; to find them all by our method would require special 3D-graphic capabilities.

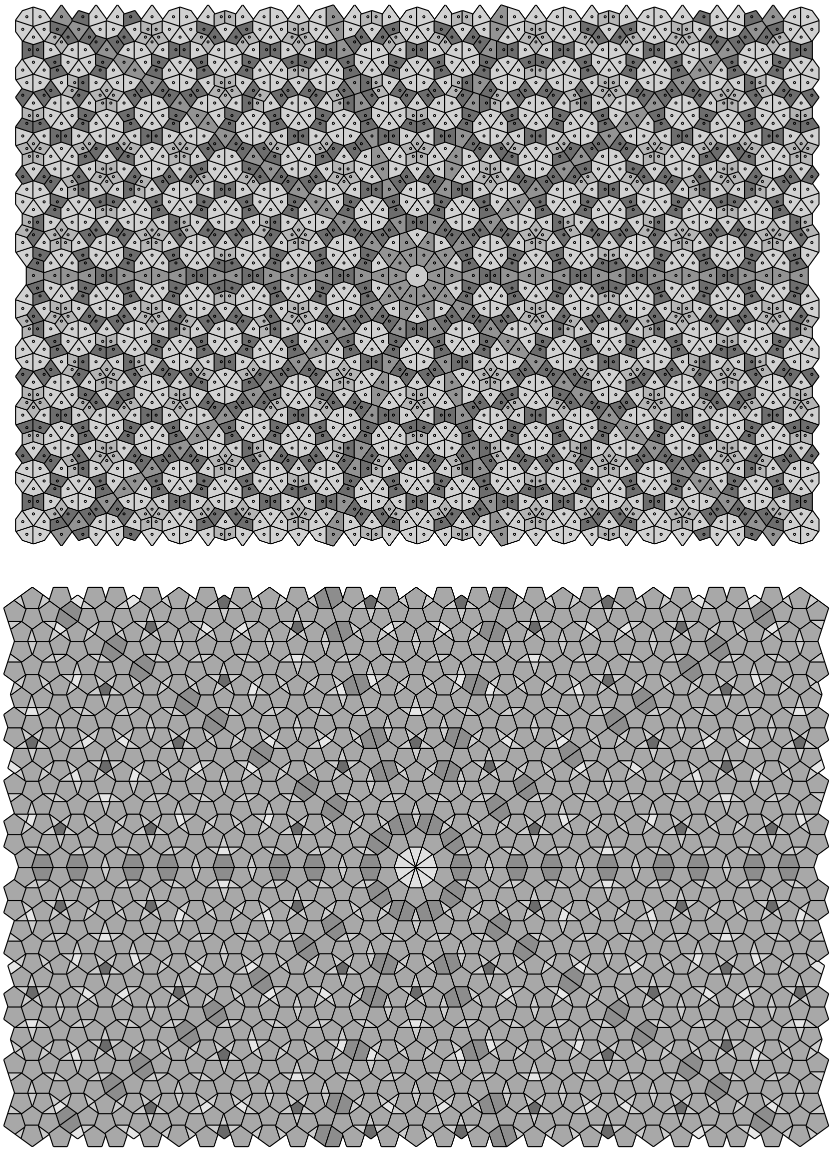


Figure 8. Voronoi and Delone tilings of a quasicrystal with decagonal acceptance window inscribed in the unit circle centred at the origin, i.e. cases VT_{14} and DT_6. Since the window is centred at the origin, its boundary has an infinite intersection with the $\mathbb{Z}[\tau]$-module M, and thus the Voronoi and Delone tilings contain tiles which appear infinitely many times, but with zero density.

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